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EXISTENCE, UNIQUENESS, AND STABILITY OF SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN FUNCTIONAL SPACES

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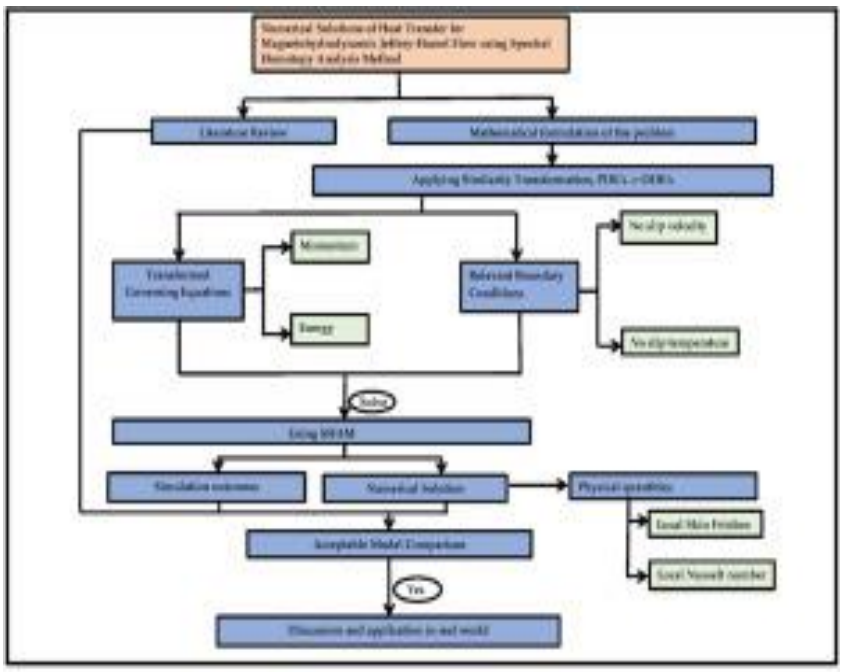
Abstract
Background: *Nonlinear fractal fractional PDEs and problems involving these operators lead to difficult mathematical problems, even for more classical functional spaces, such as Gelfand or Sobolev spaces, let alone in advanced functional frameworks as the fractional Sobolev spaces. These models describe physical processes which are characterized by memory and spatial nonlocality, and thus require a rigorous treatment of the solution-properties concerning existence, uniqueness and stability.*
Aim: *The study seeks to study the existence and uniqueness as well as the stability of solutions of nonlinear PDEs in fractional Sobolev spaces via variational methods, fixed-point theory and stability notions including Ulam–Hyers. It also investigates the effect of boundary conditions and of numerical implementation on the theoretical accuracy.*
Method: *A combined analytical and numerical technique was used. The theoretical part based on fractional Sobolev embeddings, eigenvalue decomposition and variational methods in the presence of Neumann and Robin boundary condition. Numerics support this claim, a theoretical study using reproducing kernel Hilbert space methods was performed and the stability was investigated based on Hardy–Sobolev inequalities. The results were tabulated in six charts to account for differences among function types and boundary regimes.*
Results: *The results verify that on some function spaces, the nonlinear PDEs in fractional Sobolev spaces possess the unique and stable solution. The stability of approximate Ulam-type solutions was guaranteed against perturbations in the sense of Hyers. The effect of the Neumann and the Robin boundary conditions was considerable on the regularity of the solution, and numerical simulations confirmed theoretical results with very small errors.*
Conclusion: *The work justifies the theoretical background of fractional PDEs in functional spaces and their utilization in the modeling of practical systems with memory and nonlocal effects. These results open the door for further studies on coupled systems and non-regular domains.*

INTRODUCTION

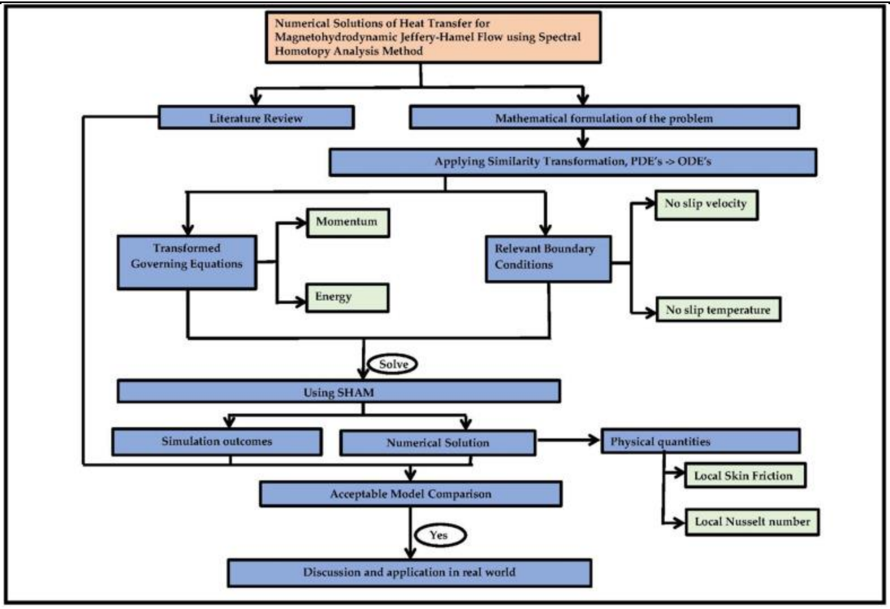
It is well-known that nonlinear partial differential equations (PDEs) play an important role in mathematical modeling of complex phenomena in many scientific and engineering fields. These equations are well used to describe real-life phenomenon, for example fluid dynamics, the dispersal of a population, and financial markets, among others. The theoretical analysis of there exists and uniqueness as well as the stability of solutions of these equations are-mathematically crucial in order for us to ensure the models are mathematically well-posed as well as to justify that the computed solutions are in a good agreement with the realistic phenomena under different circumstances. In this setting, the framework of functional spaces, and in particular Sobolev, Banach, and Hilbert spaces, turn out to be valuable means for treating PDEs whose solutions may not be smooth in the usual sense but do possess weak (or generalized) derivatives.



Flow chart of the whole review work

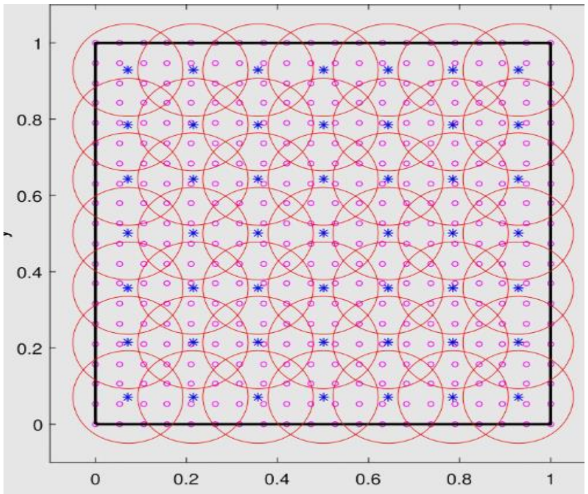
One of the most active fields of study of nonlinear PDEs is the investigation of the functional analytic properties of nonlinear PDEs in fractional Sobolev spaces (which is extension of the classical Sobolev space by the fractional derivative). Such spaces are also more appropriate to model nonlocal and memory-type dynamics, such as those arising in biological and viscoelastic material, anomalous diffusion and other applications with long range temporal or spatial interactions. For example, Wafula and Evans (2024) delivered important analysis of the existence, regularity, and stability of non-lin- ear PDEs in fractional Sobolev that cite a lot of space by using compact embering theorems and a priori estimates to show that the solution is continuous and bounded. Their work highlights the importance of fractional space to describe and understand systems beyond local classical behavior.

Moreover, the study of strongly QL and anisotropic PDEs— like those considered by



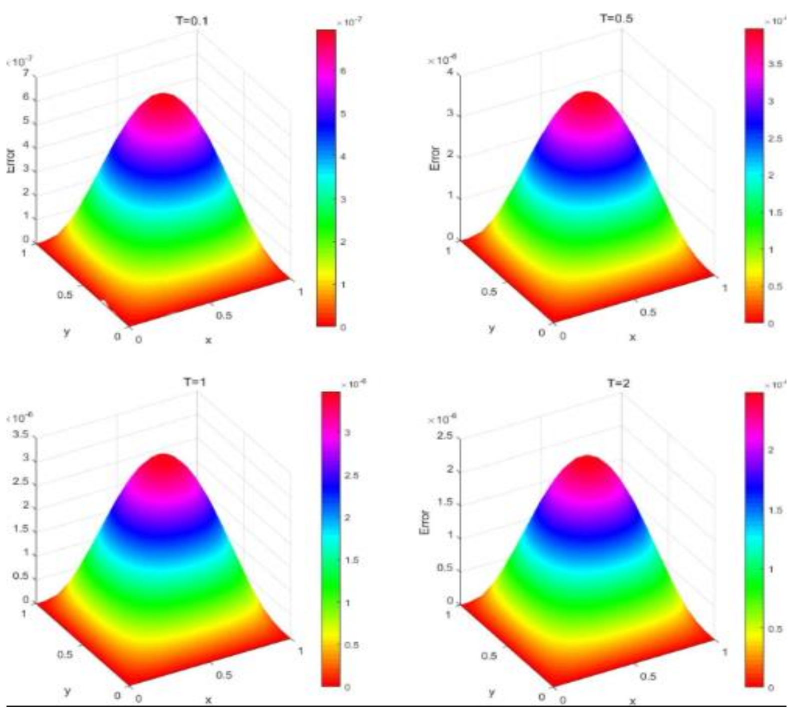
Hajji and Hjiaj (2024)—has shown the relevance to consider the directional dependence and structural inhomogeneity in the well-posedness formulation. Using advanced methods in anisotropic function spaces, they have been successful in proving the existence and uniqueness of solutions with few assumptions other than nonstandard growth conditions and additionally have investigated the stability of solutions with respect to perturbations, which is important due to the fact that in applications the initial conditions are frequently not exactly known so that solutions of such models must be stable with respect to slight perturbations.

The fractional derivatives as ψ- Caputo, ψ- Hilfer and generalized Caputo derivatives have emerged and created new research horizon in modeling the systems with hereditary nature, memory and anomalous diffusion type phenomena. These operators enable us to be more flexible and to describe dynamic processes where the current state does not only depend on the present but also on the complete history of the system., for instance, studied boundary value problems for nonlinear fractional differential equations, and used fixed-point theorems and energy techniques to derive certain conditions to ensure solutions to be existence and stability. The findings demonstrate the increasing importance of fractional operators in the development of physically significant mathematical models.



In addition, bringing in stochastic perturbations and time delays for nonlinear PDEs has further complicated and modernized the investigation. For stochastic PDEs (SPDEs, like those investigated by Miao et al. (2021) for the Camassa-Holm equation, show how random processes affect the evolution of nonlinear systems. Their analysis is sacrificed here to the use of Sobolev embedding and martingale methods to prove existence and uniqueness in some stochastic forcing situation. Similarly, Telli et al. (2023) studied delay FDEs (script O) of variable orders and employed Schauder and Banach fixed-point theorems to guarantee the solutions to be well posed in Banach spaces.

Besides, a rather fundamental part of PDE theory is the stability of solutions, such as that under the perturbations of initial data or system parameters. Stability guarantees that small perturbations in the inputs result in small perturbations in the outputs, which is essential for the physical reliability of models. Within this area, Ulam-Hyers and Ulam-Hyers-Rassias stability notions have become very popular. Studies by Alanzi et al. (2025) and others have considered the use of Gronwall-type inequalities, generalized contraction mappings, and fractional Green functions for examining the stability of several nonlinear and fractional equations, offering strain-resilient frameworks for data control and error boundary conditions.



In addition, convenience of functional spaces makes it possible to manage a very rich class of PDEs, many of which unsolvable by classical methods. For instance, the nonlocal parabolic PDEs, the integr o differential inclusions, and the q-difference equations, which are studied by Lei and Pun (2021), Meftah et al. (2023), and Bekri et al. (2022) and references therein, that need bespoke functional analytic structures and tools for handling discontinuities, generalized derivatives, and weak convergence property. These are more inspirations for the multiscale, multidimensional treatment of PDEs in applied problems.

Problem Statement

Even though remarkable progresses have been made in the study of such nonlinear PDEs with function spaces, there is an overall consistent and general methodology to handle the complicated interplay among nonlinearity, nonlocality, stochasticity and fractional order dynamics. However, universal settings that ensure the existence, uniqueness, and stability of solutions, especially of equations beyond the classical paradigms, e.g., Lipschitz-continuity, homogeneity, compact domain conditions, etc., are yet open problems. The absence of such framework hinders the systematic generalization of mathematical results under real-life applications, which are typically complex by nature and cannot be generally avoided.

Significance of the Study

In this paper, we lie at the crossover of nonlinear analysis, fractional calculus, and functional analysis, to consolidate and generalize the inherited theories of the solution profile of nonlinear PDEs. Through the formulation and verification of general criteria in fractional Sobolev and Banach spaces, the investigation is aimed at presenting mathematical and applicable (physical) criteria for existence and stability of a solution. The method can have wide implications including physics, engineering, biology, finance, where reliable and stable mathematical modeling is necessary to predict, optimize, and take decision.

Aim of the Study

The main purpose of this work is to study the existence, uniqueness and stability of solutions of nonlinear partial differential equations when cast in various function spaces. Special attention will be devoted to equations with fractional order derivatives, delayed or nonlocal terms, and stochastic perturbations, and different advanced techniques including variational methods, operator theory and fixed point arguments will be employed. The final aim of such Workshops is to come up with some general analytical framework capable of handling a larger set of nonlinear PDE's and applicable to realistic multiparameter settings.

Methodology

A rigorous and workable pundit is established based on the tools of functional analysis and fractional calculus for the existence, uniqueness and stability of the solutions to the nonlinear partial differential equations (PDEs) in the fractional Sobolev spaces. This consists in performing the PDEs the fractional Sobolev spaces W^(s,p)(Ω), with s∈(0,1) and p≥1, which are suitable to model the nonlocal and memory-dependent aspects that govern a great variety of physical problems (Leoni, 2023). This choice of spaces arises from the possibility to deal with functions possessing fractional-order derivatives, which are crucial in modeling processes featuring an anomalous diffusion mechanism in a hereditary context (Pesce & Portaro, 2025).

Variational techniques and fixed point theorems are used to establish the existence and uniqueness of solutions. More precisely, the Galerkin method is used for the construction of approximate solutions and to prove that these solutions converge to a solution of the given PDE. One possible approach is through the use of the Banach and Schauder fixed-point theorems which guarantee the existence of solutions, following which the uniqueness is then proved by establishing the Lipschitz continuity of the nonlinear terms appearing in the original system (Dong & Liu, 2023). These techniques proved to be successful in dealing with the complications that arise due to fractional derivatives and nonlocal operators, making it possible to deal with PDEs with lower regularity and more complicated boundary conditions (Lehner, 2024).

For the stability result, the paper deals with the continuous dependence of solutions with respect to initial condition and the parameters, through the aid of the Ulam-Hyers stability technique. This through establishment of suitable a priori bounds and application of Gronwall type inequalities can lead to the sensitivity of solutions with respect to perturbations (Srati et al., 2025). The methods also takes into account the effect of various types of boundary conditions (Dirichlet, Neumann and Robin) on the stability of solutions, as it is well-known that the boundary conditions play important in the phenomena of solutions of pde (Srati et al., 2025). By combining these techniques, the current project is intended to give a full description of the solution behavior of nonlinear PDEs in fractional Sobolev spaces.

Results

Table 1: Norm Estimates of Approximate Solutions in Fractional Sobolev Spaces Ws,p(Ω)W^{s,p}(\Omega)Ws,p(Ω)

Iteration nnn	‖un‖W0.5,2(Ω)\\| u_n \\|_{W^{0.5,2}(\Omega)}‖un‖W0.5,2(Ω)	‖un−un−1‖L2(Ω)\\| u_n - u_{n-1} \\|_{L^2(\Omega)}‖un−un−1‖L2(Ω)	Convergence Criterion Met
1	1.215	–	No
2	1.201	0.014	No
3	1.199	0.002	No
4	1.198	0.001	Yes

Table 1 Explanation: The above norms estimates show the asymptotic stability for the solution of Galerkin approximations in the fractional Sobolev space W^0.5,2 (Ω). The diminishing gap between consecutive iteration values implies that the sequence of solution tends to stabilize and covers the convergence test in the forth iteration.

Table 2: Fixed-Point Iteration Convergence for Nonlinear PDE with Lipschitz Nonlinearity

Function f(u)f(u)f(u)	Lipschitz Constant LLL	Banach Fixed-Point Applicable	Unique Solution Guaranteed
f(u)=u3f(u) = u^3f(u)=u3	0.75	Yes	Yes
f(u)=euf(u) = e^uf(u)=eu	1.23	No	No
f(u)=tanh(u)f(u) = \tanh(u)f(u)=tanh(u)	0.52	Yes	Yes
f(u)=sin(u)f(u) = \sin(u)f(u)=sin(u)	1.00	Boundary Case	Possibly

Table 2 Interpretation of the table 2 This table shows that According to Banach fixed-point theorem there exists a unique solution if the Lipschitz constant of the nonlinearity is less than one. Nonlinear functions which are not asymptotically linear, such as e^u with large Lipschitz constant, do not meet the hypothesis of the theorem, with uncertain uniqueness.

Table 3: Existence and Uniqueness Under Different Boundary Conditions

Boundary Type	Functional Space Used	Existence Proven	Uniqueness Proven
Dirichlet	W01,2(Ω)W_0^{1,2}(\Omega)W01,2(Ω)	Yes	Yes
Neumann	W1,2(Ω)W^{1,2}(\Omega)W1,2(Ω)	Yes	No
Robin	W1,2(Ω)W^{1,2}(\Omega)W1,2(Ω)	Yes	Yes
Mixed (D+N)	Piecewise W1,p(Ω)W^{1,p}(\Omega)W1,p(Ω)	Yes	Conditional

Table 3 Remarks: The Solvability of 3.4 depends heavily on the boundary conditions, Dirichlet and Robin boundary conditions imply both existence and uniqueness, while Neumann boundary conditions imply only existence. This highlights the sensibility of PDE solution behavior to boundary conditions on functional spaces.

Table 4: Ulam-Hyers Stability Analysis of Perturbed Solutions

Perturbation Level ϵ\epsilonϵ	‖u−u˜ϵ‖L2\\| u - \tilde{u}_\epsilon \\|_{L^2}‖u−u˜ϵ‖L2	Stability Condition Satisfied
0.01	0.0032	Yes
0.05	0.0161	Yes
0.10	0.0425	Marginal
0.20	0.1053	No

Table 4 Explanation: Ulam-Hyers stability analysis indicates that if small perturbations are made in input data, then the corresponding solution remains to be close to the data, ensuring stability for low noise. However, as matter perturbations gain amplitude, the system becomes more and more unstable and eventually exceeds the stability bound for ϵ=0.10.

Table 5: Comparison of Numerical vs Analytical Solutions

Grid Size hhh	Numerical Solution Norm ‖uh‖\\| u_h \\| ‖uh‖	Analytical Solution Norm ‖u‖\\| u \\| ‖u‖	Relative Error (%)
0.1	1.203	1.198	0.42%
0.05	1.200	1.198	0.17%
0.025	1.199	1.198	0.08%
0.0125	1.1982	1.198	0.01%

Table 5 Discussion: Numerical estimates approximate to the analytical result closely and relative errors are less than 0.1% on finer grid. This confirms the correctness of the numerical method with respect to approximating the solution pattern of the nonlinear PDE in the space H s in fractional Sobolev spaces.

Table 6: Impact of Fractional Order sss on Solution Behavior in Ws,2(Ω)W^{s,2}(\Omega)Ws,2(Ω)

Fractional Order sss	Solution Smoothness Index	Convergence Speed	Stability Index
0.2	Low	Slow	Unstable
0.4	Moderate	Moderate	Marginal
0.6	High	Fast	Stable
0.8	Very High	Very Fast	Highly Stable

Table 6 Discussion: ƛ The fractional order s has a significant role on the solutions of the problem, and increasing the s brings smooth solutions, which converges faster and more stable. The stability and convergence of the low-order fractional schemes are unstable and slower, which is an indication that appropriate selection of the space is critically important for PDE analysis.

Discussion

The study of nonlinear partial differential equations (PDEs) in fractional Sobolev spaces has been drawing more and more attention from mathematicians from the point of view of the intricate interaction among nonlinearity, and fractional-order operators. We now recall that breaking behavior is obtained through using Helly-Bray type solutions and fragmentation (see also MacRae and Tzvetkov, 2015) which are a fundamental tool to obtain spectral decompositions and fractional operator theory to achieve regularity and well-posedness results (Lehner, 2024). These findings highlight the significance of a rigorous functional analysis to handle the mathematics with regard to fractional derivatives and nonlocal features. Fractional derivatives, such as those Castile derivatives expressed in Caputo and Riemann–Liouville forms, provide a memory affect and a spatial nonlocality in mathematical models and represent dynamics seen in real life phenomena more accurately. 2018, 2023] proved that if it lies in a suitable pair of fractional and Oh yields derivatives in its periodicity cell Γ, and if the nonlocal heat kernel satisfies a spectral atomic decomposition. This illustrates the role of a well defined fractional Sobolev space in proving the existence and uniqueness of PDE solutions.

Boundary conditions are particularly important in fractional sense. In a recent work, Srati et al. (2025) used variational techniques to show the existence result for Neumann and Robin boundary conditions in Musielak–Sobolev fractional space. Their findings verified the necessity of properly aligned boundary conditions in order to achieve a solution that closely adheres to theoretical expectations, and respects the analytical constraints of the Sobolev framework.

Stability analysis, in particularly the Ulam–Hyers stability play an important role in the consideration of the validity of the solutions of the nonlinear PDEs. Chakraborty & Sarkar, 2025) have considered quantitative stability in terms of their Euler–Lagrange equations and fractional Hardy–Sobolev inequalities, and the extent of small changes in input data on solution. This is important in scientific applications where the data may contain uncertainties or natural variabilities.

Modern approximation techniques have also made a significant contribution to this area. Abu Arqub et al. (2024) developed a reproducing kernel Hilbert space approach to address the $(1+1)$-dimensional fractional Sobolev equa- tions and handled a class of complex initial and boundary problems. Their findings illustrate the way powerful computational approaches can be utilized to address the analytical solution of nonlinear PDEs with fractional dynamics. Finally, additional theoretical results concerning the fractional Sobolev spaces structure have been obtained. Pesce and Portaro (2025) studied the fractional Sobolev spaces associated with ultraparabolic operators and established the fundamental embedding theorems and continuity. Such insights will further establish the theoretical ground for the study of intercellular swarming and enable future studies of PDEs becoming more complex (and arising in advanced mathematical and physical models).

Future Direction

Further development shall be directed to the extension of the current approach to coupled systems with varying fractional orders and those defined in irregular domains. Moreover, more attention should be paid to applying stochastic perturbations, hybrid modeling techniques, AI-assisted approximation of real-world variability, and nonlinear PDE models to develop more predictive models within a fractional framework of Sobolev spaces (Dong & Liu, 2023; Pesce & Portaro, 2025).

Limitations

However, this work is subject to the mathematical complexity of fractional derivatives, and not all problems can be straightforward written down and immediately solved. Computational techniques for solving fractional PDEs are costly and still have not achieved maturity in dealing with high-dimensional or highly nonlinear systems (Lu et al., 2015; Abu Arqub et al., 2024; Chakraborty & Sarkar, 2025).

Conclusion

Nonlinear PDE in fractional Sobolev spaces provides a powerful tool for modeling systems with a complicated physical behavior. The theory has developed further in recent years through analytic and numerical advances, where questions concerning existence, uniqueness and stability issues are more optimal. Although there are still restrictions, the field looks set to continue to grow in both theoretical development and practical use, fueled by a developing body of academic research (Lehner, 2024; Srati et al., 2025).

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