Policy Research Journal
ISSN (E): 3006-7030 ISSN (P) : 3006-7022 Volume 3, Issue 3, 2025

CLOSE-TO-CONVEXITY OF q-GENERALIZED DINI FUNCTION

Muhammad Rizwan Wahala [*1], **Muhammad Nawazish Zaman** [2], **Muhammad Imran** [3], **Muhey U Din** [4], **Saima Mushtaq** [5]

[*1,2,3,5] *Faculty of Sciences, The Superior University Lahore, Pakistan*
[4] *Government Islamia Graduate College, Faisalabad, Pakistan*

[*1]rizwanwahala@gmail.com, [2]nawazishzaman5@gmail.com, [3]imran12121@outlook.com, [4]muheyudin@yahoo.com, [5]saimstar20@yahoo.com

DOI: https://doi.org/10.5281/zenodo.15107092

Keywords
Analytic function, Star like function, Close-to-Convexity, q-Generalized Dini Function

Article History
Received on 19 February 2025
Accepted on 19 March 2025
Published on 29 March 2025

Copyright @Author

Corresponding Author: *

Abstract
Geometric Function Theory (GFT) is a branch of complex analysis that focuses on the geometric properties of analytic and harmonic functions. It is closely connected to the study of special functions, which act as an important part in GFT. In this work, we investigate the close-to-convexity of the q-generalized Dini function concerning star-like functions. Additionally, several related consequences stemming from the main results are discussed.

IEER
Institute for Excellence in Education & Research

INTRODUCTION

Suppose B which show the forms of class function

$$l(\mathfrak{x}) = \mathfrak{x} + \sum_{t=2}^{\infty} a_t \mathfrak{x}^t \qquad (1)$$

within the open unit disk, the functions in question be analytical denoted as $\mathfrak{D} = \{\mathfrak{x}: |\mathfrak{x}| < 1\}$. Let C represent a subgroup of B, consisting functions of univalent defined in $\mathfrak{D}$. For $0 \le \alpha < 1$, it is possible to define the groups of star-like and close-to-convex function of order α analytically in $\mathfrak{D}$ as $C^*(\alpha) = \left\{l: l \in C \text{ and } \mathbb{R}e\left(\frac{\mathfrak{x}f'(\mathfrak{x})}{h(\mathfrak{x})} > \alpha, h \in C^*\right\}\right.$, where C^* represents the class of star-like functions [21]. In simpler terms, $C^*(0) = C^*$ which refers to the well-known class of star-like functions, and similarly, $k_h(0) = k_h$ are most close group of star-like and close-to-convex functions, in that order [19].

We now provide some fundamental concepts and explanations regarding q-calculus [18]. The definition of q-number $[t]_{\mathrm{q}}$ for $\mathrm{q} \in (0,1)$,

$$[t]_{\mathrm{q}} = \begin{cases} \dfrac{1-\mathrm{q}^t}{1-\mathrm{q}}, & t \in C, \\ \displaystyle\sum_{i=0}^{t-1} \mathrm{q}^i, & t \in N. \end{cases}$$

Also, the q-factorial $[t]_{\mathrm{q}}!$ is given by

$$[t]_{\mathrm{q}}! = 0, \qquad [t]_{\mathrm{q}}! = \prod_{i=0}^{t} [i]_{\mathrm{q}}. \qquad t \in N.$$

Assume $b, \mathrm{q} \in C$ ($|\mathrm{q}| < 1$) and $t \in \mathbb{N}_\circ = \mathbb{N} \cup \{0\}$. Then the q-shifted factorial $(b; \mathrm{q})_t$ is describe by

$$(b;\mathrm{q})_0 = 1, \quad (b;\mathrm{q})_t = \prod_{i=1}^{t}\left(1 - b\mathrm{q}^{i-1}\right), \qquad t \in \mathbb{N}.$$

Suppose $x \in C - \{-t: t \in \mathbb{N}_0\}$. Then q-Gamma function is stated by

$$\Gamma_{\mathrm{q}}(x) = \frac{(\mathrm{q};\mathrm{q})_\infty}{(\mathrm{q}^x;\mathrm{q})_\infty}(1-\mathrm{q})^{1-x}, \qquad 0 < \mathrm{q} < 1.$$

The q-derivate operator $\mathcal{D}_{\mathfrak{q}}l$ of a function l is explained in a given subgroup of $\mathcal{C}$, by

$$(\mathcal{D}_{\mathfrak{q}}l)(\mathfrak{x}) = \begin{cases} \dfrac{l(\mathfrak{x}) - l(\mathfrak{q}\mathfrak{x})}{\mathfrak{x}(1-\mathfrak{q})}, & \mathfrak{x} \neq 0, \\ l'(0), & \mathfrak{x} = 0. \end{cases} \qquad (2)$$

The given $l'(0)$ is present. So easily we can analyze from the above equation (2) that $(\mathcal{D}_{\mathfrak{q}}l)_{\lim \mathfrak{q}\to 1^-}(\mathfrak{x}) = l'(\mathfrak{x})$.

By utilizing the $\mathfrak{q}$-derivative operator $\mathcal{D}_{\mathfrak{q}}l$, the classes $S_{\mathfrak{q}}^*$ and $k_{\mathfrak{q},h}$ of $\mathfrak{q}$ -starlike and $\mathfrak{q}$ -close-to-convex functions are described as follows:

[1]: A function $l \in B$ is said to be in the set $S_{\mathfrak{q}}^*$ if

$$\left|\frac{\mathfrak{x}}{l(\mathfrak{x})}(\mathcal{D}_{\mathfrak{q}}l)(\mathfrak{x}) - \frac{1}{1-\mathfrak{q}}\right| \leq \frac{1}{1-\mathfrak{q}}, \qquad \mathfrak{x} \in \mathcal{D}, \mathfrak{q} \in (0,1). \qquad (3)$$

[2]. A function $l \in B$ is used in set $k_{\mathfrak{q},h}$ if here a star-like function h such as

$$\left|\frac{\mathfrak{x}}{h(\mathfrak{x})}(\mathcal{D}_{\mathfrak{q}}l)(\mathfrak{x}) - \frac{1}{1-\mathfrak{q}}\right| \leq \frac{1}{1-\mathfrak{q}}, \qquad \mathfrak{x} \in \mathcal{D}, \mathfrak{q} \in (0,1). \qquad (4)$$

It is stated that, when $\mathfrak{q} \to 1^-$, the sets $S_{\mathfrak{q}}^*$ and $k_{\mathfrak{q},h}$ decrease the most familiar groups S^* and k_h respectively, of star-like and close-to-convex functions [16].

In both pure and applied mathematics heavily rely on special functions [20]. In geometric function theory, these functions have made significant contributions, especially in resolving the well-known Bieberbach conjecture [17]. Researchers became interested in this application of special functions in function theory[10]. The geometric properties of several special function types are the subject of a large body of literature [14]. For example, the univalence and starlikeness of hypergeometric functions were investigated by Owa and Srivastava [3]. In order to investigate certain classes of univalent functions, Srivastava and Dziok [4,5] developed a convolution operator by employing a generalized hypergeometric function [6]

Special functions are mathematical functions that arise frequently in various areas of science and engineering, often as answers to integral or different equations [11]. They typically generalize the basic functions like trigonometric, exponential, and logarithmic functions, and play an essential role in fields such as physics, engineering, probability theory, and number theory [20].

Special functions model various physical systems, such as wave phenomena, heat conduction, and quantum mechanics. They are widely used in signal processing, control theory, and electrical engineering for solving complex system equations.

The study of special functions has seen continuous development since the 18th century, with Carl Friedrich Gauss playing a major role in shaping this ongoing effort toward a unified theory of special functions [20].

IEER Institute for Excellence in Education

1. **Q -generalized Dini function:**

Let $a \in \mathcal{R}^+$; the $\mathfrak{q}$-generalized Dini function $\psi_{v,b,c}^a(\mathfrak{x};\mathfrak{q})$ is describe by

$$\psi_{v,b,c}^a(\mathfrak{x};\mathfrak{q}) = (a-v)w_{v,b,c}(\sqrt{\mathfrak{x}};\mathfrak{q}) + \sqrt{\mathfrak{x}}w'_{v,b,c}(\mathfrak{x};\mathfrak{q}),$$

$$= \frac{1}{(\mathfrak{q};\mathfrak{q})_{k-1}}\sum_{t=0}^{\infty}\frac{(-c)^t(1-\mathfrak{q})^{\left(\frac{b-1}{2}\right)}(a+2t)\left(\frac{\mathfrak{x}}{4}\right)^{\left(t+\left(\frac{v}{2}\right)\right)}}{(\mathfrak{q},\mathfrak{q}^k;\mathfrak{q})_t}$$

because the function $\psi_{v,b,c}^a(r;\mathfrak{q})$ determined by that is not from set A, We examine the $\mathfrak{q}$-generalized Dini function in the normalized version shown below, $\psi_{v,b,c}^a(\mathfrak{x};\mathfrak{q}): \mathrm{U} \to \mathcal{C}, as$

$$\psi_{v,b,c}^a(\mathfrak{x};\mathfrak{q}) = \frac{2^v(\mathfrak{q};\mathfrak{q})_{k-1}}{a(1-\mathfrak{q})^{b-1/2}}\mathfrak{x}^{1-\left(\frac{V}{\mathfrak{x}}\right)}\varphi_{v,b,c}^a(\mathfrak{x};\mathfrak{q})$$

$$= \mathfrak{x} + \sum_{t=1}^{\infty}\zeta_t\,\mathfrak{x}^{t+1},$$

Where

$$\xi_t = \frac{(-c)^t\,(a+2t)}{4^t(\mathfrak{q},\mathfrak{q}^k;\mathfrak{q})_t}.$$

We need following lemmas to prove our main results

Lemma 01:([2]) Let $l \in A$and $D_0 = 0$,$D_1 = 1$ and (a_m) be a series of actual numbers in which

$$D_m = [m]_q a_m = \frac{a_m(1-q)}{1-q}, \quad \forall m \in N, q \in (0,1).$$

suppose,

$$1 \geq D_1 \geq D_2 \geq D_3 \geq, \ldots \geq D_m \geq, \ldots \geq 0.$$

Or

$$1 \leq D_1 \leq D_2 \leq D_3, \ldots \leq D_m \leq, \ldots \leq 2.$$

$$l(x) = x + \sum_{m=2}^{\infty} a_m x^m \in k_{q,h},$$

Where,

$$H(r) = \frac{x}{1-x}.$$

Lemma 02([6]) Consider (a_m) be a series of actual numbers in which,

$$D_m = \frac{a_m(1-q^m)}{1-q}, \quad \forall m \in N, q \in (0,1).$$

Let

$$1 \geq D_3 \geq D_5 \geq D_7, \ldots \geq D_{2m-1} \geq, \ldots \geq 0.$$

either

$$1 \leq D_3 \leq D_5 \leq D_7, \ldots \leq D_{2m-1} \leq, \ldots \leq 2.$$

At that point,

$$l(x) = x + \sum_{m=2}^{\infty} a_{2m-1} x^{2m-1} \in k_{q,h},$$

Where

$$H(r) = \frac{x}{1-x^2}.$$

IEER
Institute for Excellence in Education & Research

2. Main work:

Close-to-Convexity of q-Generalized Dini Function.

Consider

$$\psi_{v,b,c}^{a}(x; q) = x + \sum_{t=1}^{\infty} a_t \, x^{t+1} \qquad a \in \mathcal{R}^+$$

Where

$$a_t = \frac{(-c)^t \; (a+2t)}{4^t \, a(q, q^k; q)_t} = \frac{(-c)^t \; (a+2t)}{4^t \, a(q; q)_t (q^k; q)t}$$

$\because (q; q)_t \quad t \geq (1-q)^t$

$$(q^{\alpha}; q)_n \geq (1-q^{\alpha})^t$$

At this step $f(x) = \frac{x}{1+x}$

$$l(x) = x(1+x)^{-1}$$

$$l(x) = x\{1 - x + x^2 - x^3 + x^4\}$$

$$l(x) = x + \sum_{t=2}^{\infty} (-1)^{n-1} x^t$$

By Convolution

$$D(\mathfrak{x},\mathfrak{q}) = \frac{r}{1+r} \quad * \psi^{a}_{v,b,c}(\mathfrak{x};\mathfrak{q})$$
$$= \mathfrak{x} + \sum_{t=1}^{\infty} a_t\ \mathfrak{x}^{n+1}$$

where $a_t = \frac{(-c)^t\ (a+2t)}{4^t\, a\, (\mathfrak{q},;\mathfrak{q})_t(\mathfrak{q}^k;\mathfrak{q})_t}$

Theorem 3.1. Let $a \geq 0$, $c \geq 0$ and

$$8\,(1-\mathfrak{q}^2)^2\ \left(1-\mathfrak{q}^{k+2}\right) \geq 0 \quad , \mathfrak{q}\ \epsilon\ (0,1)$$

Then, $\psi^{a}_{v,b,c}(\mathfrak{x};\mathfrak{q})$ q-Generalied Dini Function is close-to-convexity regarding star-like function

$$H(r) = \frac{\mathfrak{x}}{1-\mathfrak{x}}.$$

Proof. let

$$D_m = \frac{(1-\mathfrak{q}^m)}{1-\mathfrak{q}} a_m\,, \forall\ m\ \epsilon\ \mathrm{N}\,, \mathfrak{q}\ \epsilon\ (0,1).$$

So that

$$D(\mathfrak{x};\mathfrak{q}) = \mathfrak{x} + \sum_{t=1}^{\infty} a_m\ \mathfrak{x}^m$$

$$a_m = \frac{(c)^{m-1}\ (a+2(m-1))}{4^{m-1}\, a\, (\mathfrak{q}\,;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$

$$D_m = \frac{(1-\mathfrak{q}^m)}{1-\mathfrak{q}}\ \frac{(c)^{m-1}\ (a+2(m-1))}{4^{m-1}\, a\, (\mathfrak{q}\,;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$

Put $m = 1$

$$D_1 = \frac{(1-\mathfrak{q}^1)}{1-\mathfrak{q}}\ \frac{(c)^{1-1}\ (a+2(1-1))}{4^{1-1}\, a\, (\mathfrak{q}\,;\mathfrak{q})_{1-1}(\mathfrak{q}^k;\mathfrak{q})_{1-1}}$$

$$D_1 = \frac{(1-\mathfrak{q})}{1-\mathfrak{q}}\ \frac{(c)^0\ (a+2(0))}{4^0\, a\, (\mathfrak{q}\,;\mathfrak{q})_0(\mathfrak{q}^k;\mathfrak{q})_0}$$

$$D_1 = 1$$

Put $m = 2$

$$D_2 = \frac{(1-\mathfrak{q})(1+\mathfrak{q})}{1-\mathfrak{q}}\ \frac{(c)\ \ (a+2(1))}{4\, a\, (\mathfrak{q}\,;\mathfrak{q})_1(\mathfrak{q}^k;\mathfrak{q})_1}$$

By $\mathfrak{q}$-shifted factorial definition $e \in \mathcal{R}^+\ or\ k \in \mathcal{R}^+$

Def $(e,\mathfrak{q})_t = \prod_{k=0}^{t-1}(1-e\mathfrak{q}) = (1-e)(1-e\mathfrak{q})\left(1-e\mathfrak{q}^2\right)\ldots\ldots(1-e\mathfrak{q}^{t-1})$

$$(\mathfrak{q};\mathfrak{q})_1 = (1-\mathfrak{q})$$
$$(\mathfrak{q}^k;\mathfrak{q})_1 = (1-\mathfrak{q}^k)$$
$$(\mathfrak{q};\mathfrak{q})_m = (1-\mathfrak{q})\left(1-\mathfrak{q}^2\right)\left(1-\mathfrak{q}^3\right)\ldots\ldots(1-\mathfrak{q}^m)$$
$$(\mathfrak{q}^k;\mathfrak{q})_m = \left(1-\mathfrak{q}^k\right)\left(1-\mathfrak{q}^{k+1}\right)\left(1-\mathfrak{q}^{k+2}\right)\ldots(1-\mathfrak{q}^{k+m-1})$$

$$D_2 = (1+\mathfrak{q})\ \frac{(c)\ \ (a+2)}{4\, a\, (1-\mathfrak{q})(1-\mathfrak{q}^k)}$$

$$\Rightarrow \qquad D_2 = \frac{(1+\mathfrak{q})\ (c)\ \ (a+2)}{4\, a\, (1-\mathfrak{q})(1-\mathfrak{q}^k)} \quad \leq 1.$$

We'll demonstrate it next.

$$D_{m+1} \leq D_m \quad (m\ \epsilon\ \mathrm{N}-\{1\})$$

That's what this suggests.

$$\frac{(1-\mathfrak{q}^{m+1})}{1-\mathfrak{q}}\ \frac{(c)^{m+1-1}\ (a+2(m+1-1))}{4^{m+1-1}\, a\, (\mathfrak{q}\,;\mathfrak{q})_{m+1-1}(\mathfrak{q}^k;\mathfrak{q})_{m+1-1}} \leq \frac{(1-\mathfrak{q}^m)}{1-\mathfrak{q}}\ \frac{(c)^{m-1}\ (a+2(m-1))}{4^{m-1}\, a\, (\mathfrak{q}\,;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$

$$(1-\mathfrak{q}^{m+1})\ \frac{(c)^m\ (a+2(m))}{4^m\ (\mathfrak{q}\,;\mathfrak{q})_m(\mathfrak{q}^k;\mathfrak{q})_m} \leq (1-\mathfrak{q}^m)\ \frac{(c)^{m-1}\ (a+2(m-1))}{4^{m-1}\ (\mathfrak{q}\,;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$

$$(1-\mathfrak{q}^{m+1})\ \frac{(c)^m\ (a+2m)}{4^m\ (\mathfrak{q}\ ;\mathfrak{q})_m(\mathfrak{q}^k;\mathfrak{q})_m}\ \le\ (1-\mathfrak{q}^m)\ \frac{4(c)^m\ (a+2(m-1))}{4^m\ c\ (\mathfrak{q}\ ;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$
$$(1-\mathfrak{q}^{m+1})\ \frac{(a+2m)}{(\mathfrak{q}\ ;\mathfrak{q})_m(\mathfrak{q}^k;\mathfrak{q})_m}\ \le\ (1-\mathfrak{q}^m)\ \frac{4\ (a+2(m-1))}{c\ (\mathfrak{q}\ ;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$
$$(1-\mathfrak{q}^{m+1})\ \frac{c\ (a+2m)}{4\ (1-\mathfrak{q}^m)(1-\mathfrak{q}^{k+m-1})}\ \le\ (1-\mathfrak{q}^m)\ \frac{(a+2(m-1))}{(\mathfrak{q}\ ;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$
$$\frac{(1-\mathfrak{q}^{m+1})\ c\ (a+2m)}{4\ (1-\mathfrak{q}^m)(1-\mathfrak{q}^{k+m-1})}\ \le\ \frac{(1-\mathfrak{q}^m)\ (a+2(m-1))}{1}$$

This is equivalent

$$\left(1-\mathfrak{q}^{m+1}\right).C.(a+2m)\ \le 4\,(1-\mathfrak{q}^m)^2\left(1-\mathfrak{q}^{k+m-1}\right)(a+2(m-1))$$

For $a=0$, $=c=0$, $a\ge 0$, $c\ \ge 0$.

$$\left(1-\mathfrak{q}^{m+1}\right).0.(0+2m)\ \le 4\,(1-\mathfrak{q}^m)^2\left(1-\mathfrak{q}^{k+m-1}\right)(0+2(m-1))$$
$$0\ \le 4\,(1-\mathfrak{q}^m)^2\left(1-\mathfrak{q}^{k+m-1}\right)2(m-1)$$

Put $m\ =\ 2$

$$0\ \le 4\left(1-\mathfrak{q}^2\right)^2\left(1-\mathfrak{q}^{k+2-1}\right)2(2-1))$$
$$0\ \le 4\left(1-\mathfrak{q}^2\right)^2\left(1-\mathfrak{q}^k\right)2(1))$$
$$0\ \le 4\left(1-\mathfrak{q}^2\right)^2\left(1-\mathfrak{q}^{k+1}\right).2$$
$$0\ \le 8\left(1-\mathfrak{q}^2\right)^2\left(1-\mathfrak{q}^{k+1}\right)$$
$$8\left(1-\mathfrak{q}^2\right)^2\left(1-\mathfrak{q}^{k+1}\right)\ge 0$$

Theorem 3.2: Let $a\ge 0$, $c\ge 0$ and

$$16(1-\mathfrak{q})^{2m-2}(1-\mathfrak{q})^{2m}(1-\mathfrak{q})^{k+m}(a+4m-4)\ge c^2\,(1-\mathfrak{q})^{2m+1}\,(a+4m),\quad \mathfrak{q}\epsilon(0,,1).$$

Then, the normalized the $\mathfrak{q}$-Generalized Dini function, where

$$H(r)=\frac{\mathfrak{x}}{1-\mathfrak{x}^2}$$

Proof. Consider

$$D_m=\frac{(1-\mathfrak{q}^m)}{1-\mathfrak{q}}a_m\,,\forall\,m\,\epsilon\,N\,,\mathfrak{q}\,\epsilon\,(0,1).$$

So that $D(\mathfrak{x};\mathfrak{q})=\mathfrak{x}+\sum_{t=1}^{\infty}a_m\ \mathfrak{x}^m$
where

$$a_m=\frac{(c)^{m-1}\ (a+2(m-1))}{4^{m-1}\,a\,(\mathfrak{q}\,;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$
$$D_m=\frac{(1-\mathfrak{q}^m)}{1-\mathfrak{q}}a_m\,,\forall\,m\,\epsilon\,N\,,\mathfrak{q}\,\epsilon\,(0,1).$$

So that

$$D_m=\frac{(1-\mathfrak{q}^m)}{1-\mathfrak{q}}\ \frac{(c)^{m-1}\ (a+2(m-1))}{4^{m-1}\,a\,(\mathfrak{q}\,;\mathfrak{q})_{m-1}(\mathfrak{q}^k;\mathfrak{q})_{m-1}}$$

Put $m=1$

$$D_1=\frac{(1-\mathfrak{q}^1)}{1-\mathfrak{q}}\ \frac{(c)^{1-1}\ (a+2(1-1))}{4^{1-1}\,a\,(\mathfrak{q}\,;\mathfrak{q})_{1-1}(\mathfrak{q}^k;\mathfrak{q})_{1-1}}$$
$$D_1=\frac{(1-\mathfrak{q})}{1-\mathfrak{q}}\ \frac{(c)^0\ (a+2(0))}{4^0\,a\,(\mathfrak{q}\,;\mathfrak{q})_0(\mathfrak{q}^k;\mathfrak{q})_0}$$
$$=\frac{(1-\mathfrak{q})}{1-\mathfrak{q}}\ \frac{1(a+0)}{1.a.1.1}$$
$$=\ \frac{a}{a}=\ 1$$

Put $m = 3$

$$D_3 = \frac{(1-q^3)}{1-q}\,\frac{(c)^{3-1}\ (a+2(3-1))}{4^{3-1}\,a\,(q\,;q)_{3-1}(q^k;q)_{3-1}}$$

$$D_3 = \frac{(1-q)(1+q+q^2)}{1-q}\,\frac{(c)^2\ (a+2(2))}{4^2\,a\,(q\,;q)_2(q^k;q)_2}$$

$$D_3 = (1+q+q^2)\,\frac{(c)^2\ (a+4)}{16\,a\,(q\,;q)_2(q^k;q)_2}$$

It suggest that $D_3 \le 1$

By q-shifted definition

$$(1+q+q^2)\frac{(c)^2\ (a+4)}{16\,a\,(1-q)(1-q^2)(1-q^k)(1-q^{k+1})} \le 1$$

$$\frac{(1+q+q^2)(c)^2\ (a+4)}{16\,a\,(1-q-q^2+q^3)(1-q^k-q^{k+1}+q^{2k+1})} \le 1$$

Next we will explain that.

$$D_{2m+1} \le D_{m-1} \quad (m\,\epsilon\ N-\{1\})$$

$$\frac{(1-q^{2m+1})}{1-q}\,\frac{(c)^{2m+1-1}\ (a+2(2m+1-1))}{4^{2m+1-1}\,a\,(q\,;q)_{2m+1-1}(q^k;q)_{2m+1-1}} \le \frac{(1-q^{2m-1})}{1-q}\,\frac{(c)^{2m-1-1}\ (a+2(2m-1-1))}{4^{2m-1-1}\,a\,(q\,;q)_{2m-1-1}(q^k;q)_{2m-1-1}}$$

$$\frac{(1-q^{2m+1})}{1-q}\,\frac{(c)^{2m}\ (a+2(2m))}{4^{2m}\,a\,(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})}{1-q}\,\frac{(c)^{2m-2}\ (a+2(2m-2))}{4^{2m-2}\,a\,(q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{(1-q^{2m+1})}{1-q}\,\frac{(c)^{2m}\ (a+4m)}{4^{2m}\,a\,(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})}{1-q}\,\frac{4^2(c)^{2m}\ (a+2(2m-2))}{4^{2m}\ c^2 a\,(q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{(1-q^{2m+1})(a+4m)}{4^{2m}\,a\,(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})\,4^2\ (a+2(2m-2))}{4^{2m}\,c^2 a\,(q;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{(1-q^{2m+1})(a+4m)}{4^{2m}\ (q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})\,4^2\ (a+2(2m-2))}{4^{2m}\,c^2\ (q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{(1-q^{2m+1})(a+4m)}{(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})\,4^2\ (a+2(2m-2))}{c^2\ (q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{(1-q^{2m+1})(a+4m)}{(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})\,4^2\ (a+4m-4)}{c^2\ (q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{c^2(1-q^{2m+1})(a+4m)}{(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{(1-q^{2m-1})\,4^2\ (a+4m-4)}{(q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

$$\frac{c^2(1-q^{2m+1})(a+4m)}{(q\,;q)_{2m}(q^k;q)_{2m}} \le \frac{16\,(1-q^{2m-1})\ (a+4m-4)}{(q\,;q)_{2m-2}(q^k;q)_{2m-2}}$$

By q-shifted definition

$$\frac{c^2(1-q^{2m+1})(a+4m)}{(1-q^2)(1-q^4)\dots(1-q^{2m-2})(1-q^{2m})(1-q^k)(1-q^{k+2})\dots(1-q^{k+2m-2})(1-q^{k+2m})} \le \frac{16\,(1-q^{2m-1})\ (a+4m-4)}{(1-q^2)(1-q^4)\dots(1-q^{2m-2})(1-q^k)(1-q^{k+2})\dots(1-q^{k+2m-2})}$$

$$\frac{c^2(1-q^{2m+1})(a+4m)}{(1-q^{2m})(1-q^{k+2m})} \le 16\,(1-q^{2m-1})\ (a+4m-4)$$

$$c^2(1-q^{2m+1})(a+4m) \le 16\,(1-q^{2m-1})\ (a+4m-4)\,(1-q^{2m})(1-q^{k+2m})$$

For $a = 0\ ,\ = c = 0 \qquad ,\ a \geq 0\ ,\ c\ \geq 0\,.$

$$0^2(1-\mathrm{q}^{2m+1})(0+4m) \ \leq\ 16\,(1-\mathrm{q}^{2m-1})\ (0+4m-4)\,(1-\mathrm{q}^{2m})(1-\mathrm{q}^{k+2m})$$
$$0\times(1-\mathrm{q}^{2m+1})(4m) \ \leq\ 16\,(1-\mathrm{q}^{2m-1})\ (4m-4)\,(1-\mathrm{q}^{2m})(1-\mathrm{q}^{k+2m})$$
$$0 \ \leq\ 16\,(1-\mathrm{q}^{2m-1})\ (4m-4)\,(1-\mathrm{q}^{2m})(1-\mathrm{q}^{k+2m})$$
$$0 \ \leq\ 16\ \times 4\,(m-1)\,(1-\mathrm{q}^{2m-1})\ \ (1-\mathrm{q}^{2m})(1-\mathrm{q}^{k+2m})$$
$$0 \ \leq\ 64\,(m-1)\,(1-\mathrm{q}^{2m-1})\ \ (1-\mathrm{q}^{2m})(1-\mathrm{q}^{k+2m})$$
$$64\ (m-1)\left(1-\mathrm{q}^{2m-1}\right)\,\left(1-\mathrm{q}^{2m}\right)\left(1-\mathrm{q}^{k+2m}\right)\ \geq\ 0.$$

Hence Proved.

References

Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. Complex Var. Theory Appl. **1990**, 14, 77–84.

Sahoo, S.K.; Sharma, N.L. On a generalization of close-to-convex functions. Ann. Polon. Math. **2015**, 113, 93–108.3.

Owa, S.; Srivastava, H.M. Univalent and starlike generalized hypergeometric functions. Can. J. Math. **1987**, 39, 1057–1077.

Dziok, J.; Srivastava, H.M. Classes of analytic functions associated with the generalized hypergeometric function. Appl. Math. Comput. **1999**, 103, 1–13.

Dziok, J.; Srivastava, H.M. Certain subclasses of analytic functions associated with the generalized hypergeometric function. Intg. Transf. Spec. Funct. **2003**, 14, 7–18.

Raghavendar, K.; Swaminathan, A. Close-to-convexity of basic hypergeometric functions using their Taylor coefficients. J. Math. Appl. **2012**, 35, 111–125

Askey, R. (1973). Grünbaum's inequality for Bessel functions. *Journal of Mathematical Analysis and Applications*, *41*(1), 122-124.

Almkvist, G., & Berndt, B. (1988). Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, π, and the Ladies Diary. *The American Mathematical Monthly*, *95*(7), 585-608.

Anderson, G. D., Vamanamurthy, M. K., & Vuorinen, M. (1997). Conformal invariants, inequalities, and quasiconformal maps.

Andrews, G. E., Askey, R., Roy, R., Roy, R., & Askey, R. (1999). *Special functions* (Vol. 71, pp. xvi+-664). Cambridge: Cambridge university press.

Baricz, Á. (2002). Applications of the admissible functions method for some differential equations. *Pure Mathematics and Applications*, *13*(4), 433-440.

Baricz, A. (2006). Bessel transforms and Hardy space of generalized Bessel functions. *Mathematica*, *48*(71), 127-136.

Baricz, A. (2010). Generalized Bessel functions of the first kind. *Lecture Notes in Mathematics/Springer-Verlag*.

Baricz, Á., & Baricz, Á. (2010). Geometric properties of generalized Bessel functions. *Generalized Bessel functions of the first kind*, 23-69.

Bansal, D., & Prajapat, J. K. (2016). Certain geometric properties of the Mittag-Leffler functions. *Complex Variables and Elliptic Equations*, *61*(3), 338-350.

Carlson, B. C., & Shaffer, D. B. (1984). Starlike and prestarlike hypergeometric functions. *SIAM Journal on Mathematical Analysis*, *15*(4), 737-745.

De Branges, L. (1985). A proof of the Bieberbach conjecture. *Acta Mathematica*, *154*(1), 137-152.

Din, M. U., Raza, M., Xin, Q., Yalçin, S., & Malik, S. N. (2022). Close-to-Convexity of q-Bessel–Wright Functions. *Mathematics*, *10*(18), 3322.

Din, M. U., Mushtaq, S., & Maqbool, I. (2024). Close-to-Convexity of q-Bessel Struve function. (Submitted).

Iwasaki, K., Kimura, H., Shimemura, S., & Yoshida, M. (2013). *From Gauss to Painlevé: a modern theory of special functions* (Vol. 16). Springer Science & Business Media.

Merkes, E. P., & Scott, W. T. (1961). Starlike hypergeometric functions. *Proceedings of the American Mathematical Society*, *12*(6), 885-888.

IEER Institute for Excellence in Education & Research