Policy Research Journal
ISSN (E): 3006-7030 ISSN (P) : 3006-7022 Volume 3, Issue 3, 2025

CERTAIN GEOMETRIC PROPERTIES OF MILLER-ROSS FUNCTIONS

Warda Siddique*[1], Aneeza Koser[2], Zunera Shoukat[3], Saima Mushtaq[4]

*[1,2,3,4]*Faculty of Sciences, The Superior University Lahore, Pakistan*

*[1]wardasiddique1998@gmail.com, [2]aneezaqayyum54@gmail.com, [3]saimamushtaq.fsd@superior.edu.pk, [3]zunera.shoukat.fsd@superior.edu.pk

DOI: https://doi.org/10.5281/zenodo.14982462

Keywords
Analyticity, starlikeness, convexity, Miller-Rose function

Article History
Received on 28 January 2025
Accepted on 28 February 2025
Published on 06 March 2025

Corresponding Author: *

Abstract
This article focuses on the geometric properties of normalized Miller-Ross function. We use constructive tactics to establish the conditions for close-to-convexity and find conditions under which the normalized Miller-Ross function is star-like. We also apply the starlike function $\frac{\sigma}{1-\sigma^2}$ *to establish the conditions for close-to-convexity.*

INTRODUCTION

Let A represent the widely recognized and extensively studied class of analytic functions g, which can be written as

$$g(q) = q + \sum_{s=2}^{\infty} c_\rho q^\rho , \quad q \in U.$$

In A, a one-to-one function. A function is called starlike if it maps U into a domain that is starlike in relation to the origin, and convex if it maps U onto a convex domain. Convex univalent functions in U and all-star-like functions in U are represented by S^* and C, respectively.

The generalizations of S^* and C , denoted by$S^*(\phi)$ $(star\ like)$ and $C(\phi)$ $(convem)$ of order $\phi \in [0,1]$, respectively, are as follows:

$$S^*(\phi) = \left\{ g \colon \mathrm{Re}\left(\frac{qg^{'}(q)}{g(q)} \right) > \phi ,\, q \in U \right\}$$

and

$$C(\phi) = \left\{ g \colon Re\left(1 + \frac{qg''(q)}{g(q)}{}' \right) > \phi ,\, q \in U \right\}$$

These definitions serve as the foundation for investigating the Miller-Ross function's geometric features as well as its interactions with other mathematical functions and operators.

The definition of the m (ϕ) (close-to-convex) class of order ϕ is:

$$\rho(\phi) = \left\{ g \colon \mathrm{Re}\left(\frac{q\, g^{'}\,(q)}{g\,(q)} \right) > \phi ,\, q \in U ,\, g \in S^*(0) \right\}$$

Assume that g is in A. The class T of normally real functions is thus described as follows:

$$T = \{ g \colon I\rho\ (q) I\rho\ (g(q)) > 0 ,\, q \in U \}.$$

If $g(U)$ forms a convex set in the imaginary axis direction, then a normalized univalent function f is part of c_ρ , the class of convex functions in that direction. Stated differently,

$$[m_1, m_2] \subseteq g\ (U),\ m_1, m_2 \in g\ (U),\ and\ Re\ (m_1) = Re\ (m_2).$$

If $g \in T$ and satisfies $Re\ (g'(q)) > 0$ for $q \in U$, then $g \in S^*$(starlike), as shown in [4]. The extended definition with regard to order & was introduced by Mondal and Swaminathan in [5].

The **Miller-Ross function** is demarcated by

$$E_{\eta,c}(q) = q + \sum_{K=2}^{\infty} \frac{\Gamma(\eta+1)c^{\rho-1}q^{\rho}}{\Gamma(\eta+\rho)}$$

Γ is a symbol for the Gamma function. In C, the infinite series converges completely when $a >- 1$, but not when $a =- 1$, it converges inside the open unit disk U. It is worth noting that $E_{(\eta,c)}$ is a complete function. The Miller-Ross function is an extension and adaption of fractional calculus principles, which are commonly used to solve fractional differential equations and simulate memory-dependent processes.

K.S. Miller and B. Ross proposed the function, which was intensively investigated for its applications in fractional integrals and derivatives [2]. They investigated its characteristics using fractional calculus and integral equations. The function provides a broad foundation for comprehending fractional operators and their applications in numerous scientific fields.

In particular, the Miller-Ross function bridges fractional calculus with operational methods, being crucial to the theory of partial differential equations with fractions. It offers tools for solving fractional integral equations and developing generalized transforms, with connections to the Henkel transform and Mikusiński's [1] operational calculus.

The function also finds applications in models involving anomalous diffusion, viscoelastic systems, and other complex systems described by fractional dynamics. Its entire function nature and connection with generalized Mittag-Leffler [3] functions make it a key tool in extending classical methods of solving differential equations to their fractional counterparts. It is also possible to investigate the generalizations of many functions, including the Whittaker and Array functions, as well as whole auxiliary functions, in relation to Miller-Ross functions. Specifically, the following is the relationship between the Bessel function J_{η} and the function $E_{1,\eta+1}(\frac{-q^2}{4})$.

$$J_{\eta}(q) = \left(\frac{q}{2}\right)^{\rho} E_{1,\eta+1}\left(-\frac{q^2}{4}\right)$$
$$= \sum_{s=0}^{\infty} \frac{(-1)^s q^{2s+\rho}}{2^{2s+\rho} s!\, \Gamma(s+\rho+1)}$$

This connection emphasizes the Miller-Ross function's importance in generalizing special functions as well as fractional calculus applications. See in-depth talks in fractional calculus and its applications for further information on the Miller-Ross function.

The Hadamard product, often known as the convolution, is displayed and described as:

$$(g * h)(q) = q + \sum_{s=2}^{\infty} c_s q^s, q \in U$$

Ruscheweyh [6] created the class R_q, which includes prestarlike functions of rank z, and used the concept of convolution as follows:

Let $g \in A$. Then, $g \in R_q$ if and only if:

$$\mathrm{Re}\left(\frac{g(q)}{q}\right) > 0, \qquad V \in U \, for \, q = 1,$$

$$\frac{q}{(1-q)^{2(1-q)}} * g(q) \in S^*(q), \qquad q \in U \, for \, 0 \le q < 1.$$

In particular, when we set $q = \frac{1}{2}$, then $C = R_0$ and $S^*\left(\frac{1}{2}\right) = R_{\frac{1}{2}}$. The class R_q was generalized to $R[\mu, q]$ by Sheil-Small et al. [27]. A function$g \in R[\mu, q] \; if \; g * S_{\mu} \in S^*(q)$,

where $S_{\alpha,\mu} = \frac{u}{(1-\eta)^{2-2\mu}}$, $0 \le \mu < 1$. It is informal to see that $R[q, q] = R_q$.

Since the function $M_{a,b}$ is not a member of class A, we take into consideration the modified function that follows:

$$E_{\eta,c}(q) = \rho E_{\eta,c}(q)\Gamma(\rho)$$
$$= q + \sum_{\rho=2}^{\infty} \frac{\Gamma(\eta+1)c^{\rho-1}q^{\rho}}{\Gamma(\eta+\rho)} \; \rho >- 1, Z > 0$$

We also review the minimal principle of harmonic functions and the Schwarz reflection principle:

The Meaning of the Reflection Principle:
The theory states describe that an analytic function demarcated on the upper half plane and having precise real values on the real axis may be prolonged to the intricate plane. If an analytic function is defined on W and is an arbitrary area, having precise real values on the bottom half plane singular values on its real axis, it can be extended to the Stereographic Projection of U perpendicular to P. This notation was given by my colleague Tanaka. Tanaka also stated that the function can be provided in the following notation.

$$f(q) = \overline{f(\overline{q})}$$

This formula delivers analytic continuation to the whole complex plane [5].

The Minimum Principle of Biharmonic Functions:
A bi-harmonic function U cannot have tiniest or extreme at an interior point without it is continual [11]. Over the past few years, the starlikeness and convexity of $E_{\eta,c}$ functions have been studied, and a lot of interest has been given to some special gaussian or hyperbolic functions. For additional details, see [10, 17]. Parajapat [11] is the first who studied the convexity and the starlikeness of Raza et al. [18] examined the star-likeness and convexity of the function of the first order $E_{\eta,c}$ with multiplier. They also studied Hardy spaces and the vicinity of the function in the vicinity of the convexity. Baricz et al. [19] studied the radii of gstar and starlike functions for several normalized versions of the Miller-Ross functions. The aforementioned theorems will be proved in this paper. The close-to-starlikeness was studied by Maharana et al. [20] for the close-to-convexity was investigated by Mahrana et al. [20], in starlike and suction functions relation.
The geometric characteristics of hyper geometric functions have been investigated more recently by Sangal and Swaminathan [21] using positivity approaches.
The geometric features of $E_{\eta,c}$ are the main emphasis of this study which follows the finding of [21]. By studying $E_{\eta,c}$ starlikeness, close-to-convexity, imaginary height, central steaming prestarlikeness, we wrap up our investigation. The primary instruments of our investigation are positive approaches.

We use following lemmas.
Lemma 1 [10]. Consider the sequence $\{c_s\}_{s=1}^{\infty}$ of a optimistic factual number that
1. Let $c_1 \geq 8c_2$ and $(s-1)c_s - (1+s)c_{s+1} \geq 0, \forall_s \geq 2$. Then, $g(\sigma) = \sigma + \sum_{s=2}^{\infty} c_s\sigma^s \in K$ with reverence to starlike function $\frac{\sigma}{1-\sigma^2}$.

Lemma 2 [13] If the function $g(\sigma) = \sum_{s=1}^{\infty} c_s\sigma^{s-1}$, where $c_1 = 1$ and $c_s \geq 0,\ \forall_s \geq 2$ is analytic in U, and if $\{c_s\}_{s=1}^{\infty}$ is a convex diminishing sequence, $i.e., c_{s+2} - 2c_{s+1} + c_s \geq 0$ and $c_s - c_{s+1} \geq 0, \forall_s \geq 1,$ then $g(\sigma) > \frac{1}{2}, \forall_\sigma \in$ U.

1. Main Results:
Theorem 2.1:
Let $a \geq 1, b \geq 1$, and the Miller-Ross function $E_{\eta,c}(q)$ be defined as:

$$E_{u,c}(q) = q + \sum_{K=2}^{\infty} \frac{\Gamma(u+1)c^{k-1}q^k}{\Gamma(u+k)}$$

Then, $E_{\eta,c}(q)\, \epsilon\, m$ (close-to-convex with respect to the starlike function $\frac{q}{(1+q)^2}$ if:
1. $\Gamma(a+b) \geq 8\Gamma(b)$.
2. $2\Gamma(2a+b) \geq 3\Gamma(a+b)$.

Proof: Consider

$$E_{\eta,c}(q) = Z + \sum_{\rho=2}^{\infty} c_\rho q^\rho$$

Where,

$$c_\rho = \frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)} for\ \eta \geq 1, c_1 = 1\ , \rho = 1 \qquad (1)$$

$\forall\, \rho \geq 2$, we have to show that c_ρ gratifies the premise of lemma 1. It is clear that $a \geq 1$ and $b \geq 1$ the dissimilarity $\Gamma(a+b) \geq 8\Gamma(b)$ is gratified.
Put ρ = 1 in equation (1)

$$c_1 = \frac{\Gamma(u+1)c^{1-1}}{\Gamma(u+1)}$$

$$c_1 = 1$$

Now, put ρ = 2 in equation (1)

$$c_2 = \frac{\Gamma(\eta+1)c^{2-1}}{\Gamma(\eta+2)}$$

$c_1 \geq 8c_2$ implies

$$1 \geq 8\frac{\Gamma(\eta+1)c^1}{\Gamma(\eta+2)}$$

$$\Gamma(\eta+2) \geq 8\Gamma(\eta+1)c$$

Again for $\rho \geq 2$, consider

$$(\rho-1)c_\rho - (\rho+1)\, c_{\rho+1} > 0$$

$$(\rho-1)\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)} - (\rho+1)\frac{\Gamma(\eta+1)c^{\rho-1+1}}{\Gamma(\eta+\rho+1)} > 0$$

$$(\rho-1)\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)} - \frac{(\rho+1)\Gamma(\eta+1)c^{\rho}}{\Gamma(\eta+\rho+1)} > 0$$

$$\frac{(\rho-1)}{c\Gamma(\eta+\rho)} - \frac{(\rho+1)}{\Gamma(\eta+\rho+1)} > 0$$

$$\frac{(\rho-1)\Gamma(\eta+\rho+1)-(\rho+1)c\Gamma(\eta+\rho)}{c\Gamma(\eta+\rho)\Gamma(\eta+\rho+1)} > 0$$

$$(\rho-1)\Gamma(\eta+\rho+1) - (\rho+1)c\Gamma(\eta+\rho) > 0$$

Again put $\rho = 2$

$$(2-1)\Gamma(\eta+2+1) - (2+1)c\Gamma(\eta+2) > 0$$

$$\Gamma(\eta+3) - 3c\Gamma(\eta+2) > 0$$

$$\frac{\Gamma(\eta+3)}{\Gamma(\eta+2)} > 3c$$

$$\frac{\Gamma(\eta+3)}{\Gamma(\eta+2)} - 3c > 0$$

One can easily observe that the above countenance is non-negative for $a \geq 1,\ b \geq 1$ if $2\Gamma(2a+b) \geq 3\Gamma(a+b)$. It is clear that $\{c_\rho\}_{\rho=1}^{\infty}$ satisfies Lemma 1. This completes the result.

Theorem 2.2:
Let $a \geq 1, b \geq 1, and\ the\ Miller - Ross\ function\ E_{\eta,c}(q)$ be defined as:

$$E_{\eta,c}(q) = q + \sum_{K=2}^{\infty}\frac{\Gamma(\eta+1)c^{\rho-1}q^\rho}{\Gamma(\eta+\rho)}.$$

Then

$$If\ \Gamma(a+b) > \Gamma(b)$$

and

$$\{2\Gamma(2a+b) + \Gamma(b)\}\,\Gamma(a+b) > 4\Gamma(b)\Gamma(2a+b),$$

we have

$$\Re\left(\frac{qE'_{\eta,c}(q)}{E_{\eta,c}(q)}\right) > \frac{1}{2}, \qquad q \in U = \{|q| < 1\}.$$

Proof: To obtain our result, we first prove that the sequence

$$\{c_\rho\}_{\rho=1}^{\infty} = \left\{\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)}\right\}_{\rho=1}^{\infty}$$

is decreasing.
Since

$$\Gamma\left((\eta+1)\rho+1\right) > \Gamma((\eta+1)\rho) \qquad (\forall\ \eta \geq 1, a \geq 1\ and\ b \geq 1).$$

Therefore

$$\frac{\Gamma((\eta+1)\rho+1)}{\Gamma(b)} > \frac{\Gamma((\eta+1)\rho)}{\Gamma(b)}$$

$$\frac{\Gamma(b)}{\Gamma((\eta+1)\rho)} > \frac{\Gamma(b)}{\Gamma((\eta+1)\rho+1)}$$

Now, we prove that the sequence $\{c_\rho\}_{\rho=1}^{\infty}$ convex and decreasing. For this we prove that

$$c_\rho + c_{\rho+2} - 2c_{\rho+1} \geq 0$$

$$\left(\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)}\right) + \left(\frac{\Gamma(\eta+1)c^{\rho-1+2}}{\Gamma(\eta+\rho+2)}\right) - 2\left(\frac{\Gamma(\eta+1)c^{\rho-1+1}}{\Gamma(\eta+\rho+1)}\right) \geq 0$$

$$\left(\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)}\right) + \left(\frac{\Gamma(\eta+1)c^{\rho+1}}{\Gamma(\eta+\rho+2)}\right) - 2\left(\frac{\Gamma(\eta+1)c^{\rho}}{\Gamma(\eta+\rho+1)}\right) \geq 0$$

$$\left(\frac{\Gamma(\eta+1)c^{\rho}.c^{-1}}{\Gamma(\eta+\rho)}\right) + \left(\frac{\Gamma(\eta+1)c^{\rho}.c}{\Gamma(\eta+\rho+2)}\right) - 2\left(\frac{\Gamma(\eta+1)c^{\rho}}{\Gamma(\eta+\rho+1)}\right) \geq 0$$

$$\left(\Gamma(\eta+1)c^{\rho}\right)\left(\frac{1}{c\Gamma(\eta+\rho)} + \frac{c}{\Gamma(\eta+\rho+2)} - \frac{2}{\Gamma(\eta+\rho+1)}\right) \geq 0$$

$$\left(\Gamma(\eta+1)c^{\rho}\right)\left(\frac{\Gamma(\eta+\rho+2)\Gamma(\eta+\rho+1)+c^2\Gamma(\eta+\rho)\Gamma(\eta+\rho+1)-2c(1+\rho)\Gamma(\eta+\rho+2)}{c\,\Gamma(\eta+\rho)\ \Gamma(\eta+\rho+2)\ \Gamma(\eta+\rho+1)}\right) \geq 0$$

$$\Gamma(\eta+\rho+2)\,\Gamma(\eta+\rho+1) + c^2\Gamma(\eta+\rho)\Gamma(\eta+\rho+1) - 2c\Gamma(\rho+\eta)\Gamma(\eta+\rho+2) \geq 0$$

$$\Gamma(\eta+\rho+2)\,\Gamma(\eta+\rho+1) \geq 2c\Gamma(\eta+\rho)\Gamma(\eta+\rho+2) - c^2\Gamma(\eta+\rho)\Gamma(\eta+\rho+1) \qquad (1)$$

Put ρ =1 in (1)

$$\Gamma(\eta+3)\,\Gamma(\eta+2) \geq 2c\Gamma(\eta+1)\Gamma(\eta+3) - c^2\Gamma(\eta+1)\Gamma(\eta+2)$$

$$\Gamma(\eta+3)\,\Gamma(\eta+2) \geq c\Gamma(\eta+1)[2\Gamma(\eta+3) - c\Gamma(\eta+2)]$$

$$\frac{1}{c\Gamma(\eta+1)} \geq \left(\frac{2\Gamma(\eta+3) - c\Gamma(\eta+2)}{\Gamma(\eta+3)\Gamma(\eta+2)}\right)$$

$$\frac{1}{c\Gamma(\eta+1)} \geq \left(\frac{2\Gamma(\eta+3)}{\Gamma(\eta+3)\Gamma(\eta+2)} - \frac{c\Gamma(\eta+2)}{\Gamma(\eta+3)\Gamma(\eta+2)}\right)$$

$$\frac{\Gamma(\eta+2)\Gamma(\eta+3)}{c\,\Gamma(\eta+1)} \geq 2\Gamma(\eta+3) - c\Gamma(\eta+2)$$

$$\frac{\Gamma(\eta+2)\Gamma(\eta+3)}{c\,\Gamma(\eta+1)} - c\Gamma(\eta+2) \geq 2\Gamma(\eta+3)$$

Which shows that $\{c_\rho\}_{\rho=1}^{\infty}$ is a convex decreasing sequence. Now, from lemma 2 $\{c_\rho\}_{\rho=1}^{\infty}$ satisfy $\Re\left(\sum_{\rho=1}^{\infty} c_\rho Z^{\rho-1}\right) > \frac{1}{2}$, for all $Z \in U$. Therefore, $\Re\left\{\frac{E_{\eta,c}(q)}{Z}\right\} > \frac{1}{2}, for\ Z \in U$

Hence the result follows.

Theorem 3:

Let $a \geq 1, b \geq 1,$ *and the* Miller − Ross function $E_{a,b}(q)$ be defined as:

$$E_{\eta,c}(q) = q + \sum_{K=2}^{\infty} \frac{\Gamma(\eta+1)c^{\rho-1}q^{\rho}}{\Gamma(\eta+\rho)}$$

If:

1. $\Gamma(a+b) > 2\Gamma(b),$
2. $\{2\Gamma(2a+b) + 3\Gamma(b)\}\Gamma(a+b) > 8\Gamma(b)\Gamma(2a+b),$

then $E_{\eta,c}(q) \in S^*$

$$\Re\left(\frac{qE'_{\eta,c}(q)}{E_{\eta,c}(q)}\right) > 0 \qquad \forall q \in \mathbb{U}\ \{|q| < 1\}$$

Proof:

To prove that $E_{\eta,c} \in S^*$, we show that $\{\rho c_\rho\}$ and $\{\rho c_\rho - (\rho+1)c_{\rho+1}\}$ both are non-increasing. Since $c_\rho \geq 0$ for $E_{\eta,c}(q)$ under the given conditions. So, consider

$$\{\rho c_\rho - (\rho+1)c_{\rho+1}\} > 0$$

$$\rho\left(\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)}\right) - (\rho+1)\left(\frac{\Gamma(\eta+1)c^{\rho-1+1}}{\Gamma(\eta+\rho+1)}\right) > 0$$

$$\frac{m}{c\Gamma(\eta+\rho)} > \frac{(\rho+1)}{\Gamma(\eta+\rho+1)}$$

$$\frac{m}{c\Gamma(\eta+\rho)} - \frac{(\rho+1)}{\Gamma(\eta+\rho+1)} > 0$$

$$\frac{k\Gamma(\eta+\rho+1) - c(\rho+1)\Gamma(\eta+\rho)}{c\Gamma(\eta+\rho)\Gamma(\eta+\rho+1)} > 0$$

Put the value ρ =1

$$\Gamma(\eta+2) - c(1+1)\Gamma(\eta+1) > 0$$

$$\Gamma(\eta+2) > 2c\Gamma(\eta+1).$$

Now

$$(\rho+2)c_{\rho+2} - 2(\rho+1)c_{\rho+1} + \rho c_\rho > 0$$

$$(\rho+2)\left(\frac{\Gamma(\eta+1)c^{\rho-1+2}}{\Gamma(\eta+\rho+2)}\right) - 2(\rho+1)\left(\frac{\Gamma(\eta+1)c^{\rho-1+1}}{\Gamma(\eta+\rho+1)}\right) + \rho\left(\frac{\Gamma(\eta+1)c^{\rho-1}}{\Gamma(\eta+\rho)}\right) > 0$$

$$(\rho+2)\left(\frac{\Gamma(\eta+1)c^{\rho}.c}{\Gamma(\eta+\rho+2)}\right) - 2(\rho+1)\left(\frac{\Gamma(\eta+1)c^{\rho}}{\Gamma(\eta+\rho+1)}\right) + \rho\left(\frac{\Gamma(\eta+1)c^{\rho}}{c\,\Gamma(\eta+\rho)}\right) > 0$$

$$\Gamma(\eta+1)c^{\rho}\left(\frac{c(\rho+2)}{\Gamma(\eta+\rho+2)} - \frac{2(\rho+1)}{\Gamma(\eta+\rho+1)} + \frac{k}{c\Gamma(\eta+\rho)}\right) > 0$$

Put ρ =1

$$\Gamma(\eta+1)c\left(\frac{3c}{\Gamma(\eta+3)} - \frac{2(2)}{\Gamma(\eta+2)} + \frac{1}{c\Gamma(\eta+1)}\right) > 0$$

$$c\,\Gamma(\eta+1)\left(\frac{3c}{\Gamma(\eta+3)}-\frac{2(2)}{\Gamma(\eta+2)}+\frac{1}{c\Gamma(\eta+1)}\right) > 0$$

$$\frac{3c^2\Gamma(\eta+2)\Gamma(\rho+1)-4\Gamma(\eta+3)c\Gamma(\eta+1)+\Gamma(\eta+3}{\Gamma(\eta+3)\Gamma(\eta+2)c\Gamma(\eta+1)} > 0$$

$$3c^2\Gamma(\eta+2)\Gamma(\eta+1)-4c\Gamma(\eta+3)\Gamma(\eta+1)+\Gamma(\eta+3)\Gamma(\eta+2) > 0$$

$$3c^2\Gamma(\eta+1)\Gamma(\eta+3) > \frac{4c\Gamma(\eta+3)\Gamma(\eta+1)}{\Gamma(\eta+2)}$$

$$3c^2\Gamma(\eta+1)\Gamma(\eta+3)-\frac{4c\Gamma(\eta+3)\Gamma(\eta+1)}{\Gamma(\eta+2)} > 0$$

Then $E_{\eta,c}(q) \in S^*$

$$\Re\left(\frac{qE'_{\eta,c}(q)}{E_{\eta,c}(q)}\right) > 0 \qquad \forall\, q \in \mathbb{U}\ \{|q|<1\}$$

Hence the result is proved.

REFERENCE:

1. Luchko, Y. On the asymptotics of zeros of the Miller-Ross function. Z. Anal. Anwend. **2000**, 19, 597–622.
2. Gorenflo, R.; Mainardi, F.; Srivastava, H.M. Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. In Proceedings ZIII International Colloquium on Differential Equations, Plozdiz 1997; Bainoz, D., Ed.; ZSP: Utrecht, the Netherlands, 1998; pp. 195–202.
3. Luchko, Y. On some new properties of the fundamental solution to the multi-dimensional space- and time-fractional diffusion wave equation. Mathematics **2017**, 5, 76.
4. Mainardi, F.; Pagnini, G. The role of the Fox–Miller-Ross functions in fractional sub-diffusion of distributed order. J. Comput. Appl. Math. **2007**, 207, 245–257.
5. Ruscheweyh, S. Coefficient conditions for starlike functions. Glasgow Math. J. **1987**, 29, 141–142.
6. Mondal, S.R.; Swaminathan, A. On the positivity of certain trigonometric sums and their applications. Compute. Math. Appl. **2011**, 62, 3871–3883.
7. Ruscheweyh, S. Convolutions in Geometric Function Theory; Les Presses de l'Unizersité de Montréal: Montreal, QC, Canada, 1982; Volume 83.
8. Sheil-Small, T.; Silverman, H.; Silvia, E. Convolution multipliers and starlike functions. J. Anal. Math. **1982**, 41, 181–192.
9. Duren, P.L. Univalent Functions, Grundlehren der Mathematischen Wissenschaften; Band 259; Springer: New York, NY, USA;Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
10. Remmert, R. Theory of Complex Functions. Graduate Texts in Mathematics; Readings in Mathematics; Springer: New York, NY, USA,1991; Volume 122.
11. Cekim, B.; Shehata, A.; Srivastava, H.M. Two-sided inequalities for the Struze and Lommel functions. Quaest. Math. **2018**, 41,985–1003.
12. Prajapat, J.K. Certain geometric properties of the Miller-Ross functions. Integral Transform. Spec. Function **2015**, 26, 203–212.
13. Rehman, M.S.U.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, B.; Khan, N. Partial sums of generalized q-Mittag–Leffler functions. AIMS Math. **2020**, 5, 408–420.
14. Mustafa, N. Geometric properties of normalized Miller-Ross functions. Math. Compute. Appl. **2016**, 21, 14.
15. Srivastava, H.M.; Selzakumaran, K.A.; Purohit, S.D. Inclusion properties for certain subclasses of analytic functions defined by using the generalized Bessel functions. Malaya J. Mater. **2015**, 3, 360–367.
16. Srivastava, H.M.; Bansal, D. Close-to-convexity of a certain family of q-Mittag–Leffler functions. J. Nonlinear Zar. Anal. **2017**, 1,61–69.
17. Tang, H.; Srivastava, H.M.; Deniz, E.; Li, S.-H. Third-order differential super ordination involving the generalized Bessel functions. Bull. Malays. Math. Sci. Soc. **2015**, 38, 1669–1688.
18. Saliu, A.; Noor, K.I.; Husain, S.; Darus, M. On Bessel Functions Related with Certain Classes of Analytic Functions with respect to

Symmetrical Points. J. Math. **2021**, 2021, 6648710.

18. Raza, M.; Din, M.U.; Malik, S.N. Certain geometric properties of normalized Miller-Ross functions. J. Funct. Spaces **2016**, 2016, 1896154.
19. Baricz, A.; Toklu, E.; Kadiolu, E. Radii of starlikeness and convexity of Miller-Ross functions. Math. Common. **2016**, 23, 97–117.
20. Maharana, S.; Prajapat J.K.; Bansal, D. Geometric properties of Miller-Ross function. Math. Bohme. **2018**, 143, 99–111.
21. Sangal, P.; Swaminathan, A. Starlikeness of Gaussian hyper geometric functions using positivity techniques. Bull. Malays. Math.Sci. Soc. **2018**, 41, 507–521.